Introduction to generalized functions

Tracking the singularities

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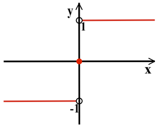
Texas Topology and Geometry Conference

Outline

Domain of operators

- M_ρ Multiplication by $\rho \in L^\infty(\mathbb{R})$



- *Question* What is $M_\rho(i\frac{d}{dx})$?

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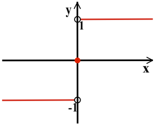

- *Question* What is $M_\rho(i\frac{d}{dx})$?
- M_ρ does not preserve Domain $i\frac{d}{dx}$.

Distributional kernals

- Composing the two operators $T = M_\rho \circ i\frac{d}{dx}$

$$
\begin{array}{ccc}
\mathcal{C}_c^\infty(\mathbb{R}) & \xrightarrow{M_\rho} & \mathcal{E}'(\mathbb{R}) \\
{\scriptstyle i\frac{d}{dx}} \uparrow & \overset{T}{\nearrow} & \\
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- *Schwartz kernel Theorem*: $\ker(T) \in D'(\mathbb{R} \times \mathbb{R})$
- Singularities of $\ker(T)$ contain much information on T

Distributions

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- Common Domain for all order k operators.
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- Frechét space

$$\mathcal{C}^\infty(M) := \bigcap_k H^k(M)$$

Distributions are the dual $D'(M) := \mathcal{C}^\infty(M)'$

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- $E \to M$ a Hermitian vector bundle
  $D : \Gamma^\infty(M : E) \to \Gamma^\infty(M : E)$

Elliptic Regularity

D Laplace operator on M

- Exist ϕ_n, λ_n

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  - First term in expansion of e^{-tD}

Spectral Theory

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- F a function on $\mathbb{R}$
- Functional Calculus $F \to F(D)$

$$F(D)\phi_n := F(\lambda_n)\phi_n.$$

  - Smoothing operator $\Psi^{-\infty}(M)$:
    - $T : D'(M) \to \mathcal{C}^\infty(M)$
    - T has a smooth kernel
  - Implecation of Weyl's Law

$$F \in S(\mathbb{R}) \Longrightarrow F(D) \in \Psi^{-\infty}(M)$$

Approximate units: Functoriality

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- $F(x) \in S(\mathbb{R})$
- $F(0) = 1$
- $u \in D'(M)$

$$F_\varepsilon(D)u \to u \quad \text{in } D'(M).$$

  - *Question* What properties of u can be recovered from the approximation $F_\varepsilon(D)u$?

Zeroth model of generalized functions

- $\tilde{\mathcal{G}}(M) = \{\varepsilon \to \gamma_\varepsilon \in \mathcal{C}^\infty(M)\}$
  Generalized functions= Curves in a Frechét space.
- A distribution u represented by $F_\varepsilon(D)u$

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- Drawbacks
  - non-unique hence non cannonical representation of distribution
- Control asymptotic growth of the curve γ_ε
  Historically Guillemin-Sternberg in ‘Geometric Asymptotics’

Asymptotics

- X a Frechét space.
- $\mathcal{G}^\infty(X)$ curves $(0,1) \ni \varepsilon \to \gamma_\varepsilon \in X$ such that There exists a fixed N in $\mathbb{Z}$

$$\text{any seminorm } \rho \to \rho(\gamma_\varepsilon) \sim O(\varepsilon^N).$$

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Theorem

If Schwartz function $F \equiv 1$ near the origin

$$F_\varepsilon(D)(D'(M)) \cap \mathcal{G}^\infty(M) = F_\varepsilon(D)(\mathcal{C}^\infty(M)).$$

Wavefront Set

- Wavefront sets= singularity directions of a distribution
- P pseudo-differential operator order 0.
- char $P \subset S^*M := \sigma_P^{-1}\{0\}$

$$WF(u) := \bigcap_{Pu \in \mathcal{C}^\infty(M)} \operatorname{char} P.$$

- generalize this to curves in $\mathcal{C}^\infty(M)$

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Theorem

Set $T_\varepsilon = F_\varepsilon(D)$ for $F \equiv 1$ near the origin:

$$WF_g(T_\varepsilon u) = WF(u).$$

Colombeau Algebra

- To summarize
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Set

$$M := \{\gamma_\varepsilon | \rho(\gamma_\varepsilon) \sim O(\varepsilon^m)\ m \text{ depands on seminorm } \rho\}$$
$$N := \{\gamma_\varepsilon | \rho(\gamma_\varepsilon) \sim O(\varepsilon^m)\ \forall\ m\ \}$$

Imposibility result

- There is no commutative algebra A
- with a linear embedding $\tau : D'(M) \to \mathcal{A}$
  such that $\tau|_{\mathcal{C}^k(M)}$ is an algebra morphism. $k = 0, 1, 2 \dots$
- Noted by Schwarts in an attempt to introduce nonlinear structure on $D'(M)$.

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However:

$$F_\varepsilon(D) : D'(M) \to M/N$$

is algebra isomorphsim restricted to $\mathcal{C}^\infty(M)$.

- $\mathcal{G}(M) := M/N$ Colombeau algebra.

Frechét Techiniques

- More canonical representations are possible:
- $X = \Psi^{-\infty}(M)$ and $Y = \mathcal{C}^{\infty}(M)$

$$D'(M) \ni u \rightarrow \Theta_u : X \rightarrow Y$$
$$\Theta_u(T) := T(u)$$

'pause

Theorem

There exists Frechét grading on $\Psi^{-\infty}(M)$ and $\mathcal{C}^{\infty}(M)$ such that

$$u \in H^s(M) \leftrightarrow \Theta_u \textit{ has tameness } \; - k$$

Modelling geophysics

- M be the Earth, S the surface of earth.
- $p(x)$ be inverse of speed of propagation of material waves at a point inside M
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$$(p(x)\partial_t^2 + \Delta)u(x,t) = \delta_0(x,t)$$
$$u(x,t) = 0 \ t < 0$$

- Find inverse of the operator A

$$A(p(x)) = u(x,t)_{|S}$$

- The linearization DA is a Fourier Integral operator when $p(x)$ is smooth

Geodesics in distributional meterics

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- What are the geodesics?
- Steinbauer-Vickers, Kunzinger-Steinbauer-Vickers at al Provide sensible notiona of geodesic equations for distributional Lorentzian meterics