Equivariant non-commutative residue.

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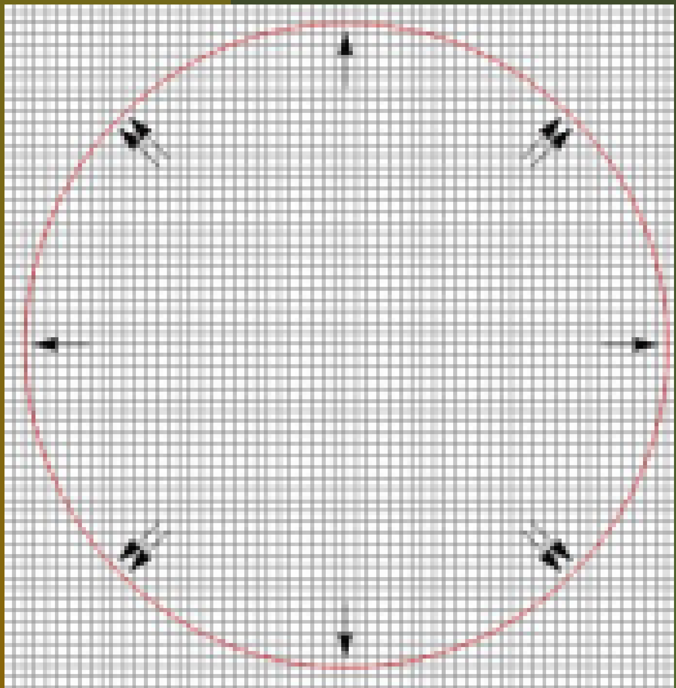
Weyl's Law

$$N_\Delta(\lambda) \sim C_n \lambda^{\frac{n}{2}}$$

Example

Flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$

- Eigenvalues $4\pi^2|\sigma|^2$ For latice points $\sigma \in \mathbb{Z}$.
- Eigenfunctions $f_\sigma(x) = e^{2\pi i\langle\sigma,x\rangle}$
- Count latice points asymptotically in spheres



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For an irreducible representation π of Γ

$$N_{\pi,\mathcal{D}}(\lambda) := \sum_{\lambda_i < \lambda} \text{“multiplicity of } \pi \text{ in } V_i\text{.”}$$

$$= \langle \pi, V_i \rangle$$

Equivariant Weyl's Law

- Usual Weyl's law

$$N_{\mathcal{D}}(\lambda) \sim C\lambda^n$$

- Action of Γ on M be faithful

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- $\mathcal{S}_n$ symmetric group on n letters
- $\mathcal{S}_n$ acts on $\mathbb{T}^n$ =permuting the circles.

$$\tau \in \mathcal{S}_n \qquad \tau(f_\sigma(x)) = f_{\tau(\sigma)}(x).$$

- Most eigenspace $\longrightarrow$ Regular Representation.

Comparison Weyl's Law

- Γ action on M faithful
- π_j be irreducible representations

$$N_{\pi_1,\pi_2,\mathcal{D}}(\lambda) := \sum_{\lambda_i \leq \lambda} \langle V_{\lambda_i}, \pi_1 - \pi_2 \rangle.$$

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- Relative dim for π_j

$$k = max_{g\in\Gamma}\{\dim(M^g)|\chi_{\pi_1}(g) \neq \chi_{\pi_2}(g)\}$$

.

- Then asymptotically

$$N_{\pi_1,\pi_2,\mathcal{D}}(\lambda) \simeq C\lambda^k.$$

Pseudodiferential operators

If M is closed manifold then

- $\Psi^{\infty}(M)$ is an algebra.
- $\Psi^{-\infty}(M)$ is an ideal in $\Psi^{\infty}(M)$.

The algebra of complete symbols is

$$\mathcal{A}(M) := \frac{\Psi^{\infty}(M)}{\Psi^{-\infty}(M)}$$

. If $A \in \mathcal{A}(M)$ is locally given by symbol expansion

$$a(x,\xi) \sim a_m(x,\xi) + a_{m-1}(x,\xi) + \dots$$

Noncommutative Residue

Seeley: For $z \in \mathbb{C}$ the operator $\mathcal{D}^z$ is a ΨDO of order z.

For $A \in \Psi^\infty(M)$ the function $\zeta_A(z) := \mathrm{Tr}(A\mathcal{D}^{-z})$

- Is holomorph on $\mathrm{Re}(z) > d = order(A) + n$.
- Extends meromorphically to $\mathbb{C}$ with possible simple poles at $d, d-1, d-2 \ldots$.

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NCR is a linear map $\tau : \mathcal{A}(M) \to \mathbb{C}$

$$\tau(A) := \mathrm{res}_{z=0}\, \zeta_A(z)$$

$\tau(A)$ is unique (up to a const.) trace on $\Psi^\infty(M)$. That is

$$\tau(AB) = \tau(BA) \qquad A, B \in \Psi^\infty(M).$$

Local Properties

If $A \in \Psi^\infty(M)$ is locally given by symbol expansion

$$a(x,\xi) \sim a_m(x,\xi) + a_{m-1}(x,\xi) + \dots$$

defines a density on M

$$Wres_A(x) := \int_{|\xi|=1} a_{-n}(x,\xi) d\xi |dx|$$

and

$$\tau(A) = C \int_M Wres_A(x).$$

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τ generator of $\mathrm{HC}^0(\mathcal{A}(M))$

Equivariant Residues

- Γ acts on $M \to \Gamma$ acts on $\mathcal{A}(M)$
- Pull back of operators by diffeomorphism

$$g.\mathcal{D}(f) := g\mathcal{D}(g^{-1}f) \qquad \forall f \in \mathcal{C}^{\infty}(M).$$

- Cross -product

$$\mathcal{A}_{\Gamma}(M) := \mathcal{A}(M) \rtimes \Gamma := \{ \sum_{g \in \Gamma} A_g g : \; A_g \in \mathcal{A}(M) \}.$$

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- Equivariant NCR $\Longrightarrow$ Traces on $\mathcal{A}_\Gamma(M)$

Zeta functions

- $\mathcal{D}$ be a positive order one elliptic Γ invariant
- For A in $\Psi^\infty(M)$ and any group element g,

$$\zeta_{g,A}(z) := \mathrm{Tr}(\mathcal{D}^{-z} A g).$$

- $k_g = \dim(M^g)$

Zeta functions

- $\mathcal{D}$ be a positive order one elliptic Γ invariant
- For A in $\Psi^\infty(M)$ and any group element g,

$$\zeta_{g,A}(z) := \mathrm{Tr}(\mathcal{D}^{-z}Ag).$$

- $k_g = \dim(M^g)$
- $\zeta_{g,A}(z)$ holomorphic on the half plane $Re(z) > d,\ d = k_g + order(A)$.
- Meromorphic extension possible simple poles at $z = d, d-1, d-2, \ldots$.

The Traces

- $\langle\gamma\rangle$ conjugacy classes
- Traces on $\mathcal{A}_\Gamma(M)$

$$\mathrm{Tr}_R^{\langle\gamma\rangle}\Big(\sum_{g\in\Gamma} A_g g\Big) := \sum_{g\in\langle\gamma\rangle} \mathrm{res}_{z=0}\, \zeta_{g,A}(z)$$

Local Properties

- Let $\mathcal{U}(x_1, x_2)$ be normal coordinates $x \in M^g$
- A symbol $a(x,\eta) \sim a_m(x,\eta) + a_{m-1}(x,\eta) + \dots$
- There is a phase function f such that

$$W_{M^g}(A) := \left(\sum_{j=0}^{m} \int_{S^k(\eta_1)} C \langle f''^{-1}_{(x_1,\eta_1)} D^\perp, D^\perp \rangle^j a_{-k_g+j}(x,\eta \right.$$

$$dS^{k_g}(\eta_1)|dx_1|$$

$$\mathrm{res}_{z=0}\, \zeta_{g,A}(z) = \int_{M^g} W_{M^g}(A).$$

Applications

- We say that a positive invariant operator $\mathcal{D}$ is π-measurable

$$\operatorname{Tr}^+_\pi(\mathcal{D}) := \lim_{N\to\infty} \frac{\sum_{i\le N} \mu_i}{\operatorname{Log} N} < \infty$$

$$= \lim_{N\to\infty} \frac{\sum_{\sum m(\pi)_j \le N} m(\pi)_j \lambda_j}{\operatorname{Log} N} < \infty$$

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- Equivariant Connes Trace formula

$$Tr^+_\pi(\mathcal{D}) = \mathrm{Tr}_{\mathrm{R}}(\mathcal{D}) = Tr^+(\mathcal{D}).$$